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基于 Parzen 窗的高阶统计量特征降维方法

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摘　要:高阶统计量通常能比低阶统计量提取更多原数据的信息,但是较高的阶数带来了较高的时间复杂度. 基于 Parzen 窗估计构造了高阶统计量,通过论证得出:对于所提出的核协方差成分分析(KCCA)方法,通过调节二阶统计量广义 D-vs-E 的参数就能够达到整合高阶统计量的目的,而无需计算更高阶统计量. 即核协方差成分分析方法能够对高阶统计量的特征降维的同时,又不增加计算复杂性.

关键词:核协方差成分分析;高阶统计量;Parzen 窗;特征降维
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Feature reduction of high-order statistics based on Parzen window

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Abstract: The high-order statistics method can often extract more information regarding original data than a low-order statistics; yet in the meantime create higher time complexity. The high-order statistics methods were constructed by utilizing estimation based on Parzen window. It was revealed that the kernel covariance component analysis (KCCA) method proposed earlier by the researchers, contained useful information on the high-order statistics and could be obtained by only adjusting the parameters of the proposed generalized D-vs-E. Also based on the second order statistics, the heavy computational burden about the high-order statistics can be avoided. That is to say, the KCCA method can accomplish the feature reduction of high-order statistics without increasing its computational complexity.

Keywords: KCCA; higher-order statistics; Parzen window; feature reduction

高阶统计量方法[1-14]是近几年国内外信号处理领域内的一个前沿课题,它往往比二阶统计量包含更多更丰富的信息,并广泛应用于模式识别、信号检测、分类等问题,人们有可能从高阶统计量获得信号的显著分类特征. 但是在本文介绍的一类基于 Parzen 窗的特征降维方法中,高阶统计量并没有提供更多信息,而是与二阶统计量提供的信息相当,因此只需要使用二阶统计量就能达到这类方法所能达到的最好效果.

R. Jenssen 提出的 KECA(kernel entropy component analysis)方法[15]所采用的统计量只考虑了 Renyi 熵,即数据集的平均向量的欧几里德长度,所以 KECA 也可以看作是降维前后核特征空间的数据平均向量的欧几里德长度变化的最小化问题. 而 KCCA(kernel covariance component analysis)方法[16]则是基于协方差矩阵来构造统计量 D-vs-E(densities-vs-entropy),这种方法要求降维前后的 D-vs-E 尽量接近. 因为 D-vs-E 不仅包含了 Renyi 熵,更包含了散度变小时所有样本的概率密度和,这是 KECA 方法中所没有的. 使用二阶统计量不仅能和 KECA 一样揭示出数据的结构,而且增强了这种降维方法对核参数选择的鲁棒性. 由此考虑到高于二阶的高阶统计量会不会包含更多的信息,从而使此类降维方

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法具有更好的性能和优势？由这个问题引出了本文对高阶统计量的探讨、分析和证明.

1 核熵成分分析(KECA)

1.1 KECA 的定义

KECA 是由 R. Jenssen 提出的一种数据转换和降维方法[15]，为了便于阅读，以下采用与文献[15]相同的数学符号. KECA 的提出基于 2 个概念：一个是 Renyi 熵[17]：

$$H(p) = -\log V(p) = -\log\int p^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x};$$

另一个是 Parzen 窗密度估计[18]：

$$\hat{p}(\boldsymbol{x}) = \frac{1}{N}\sum_{\boldsymbol{x}_t\in D} k_\sigma(\boldsymbol{x},\boldsymbol{x}_t).$$

式中：$D=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_N\}$. 由于 $k_\sigma(\boldsymbol{x},\cdot)$ 的形状对 Parzen 窗密度估计的影响并不大[19-20]，为了便于分析和计算，可以采用核参数为 σ 的高斯核函数.

$V(p)$ 可以用 $\hat{p}(\boldsymbol{x})$ 近似地表示为

$$\hat{V}(p) = \frac{1}{N}\sum_{\boldsymbol{x}_t\in D}\hat{p}(\boldsymbol{x}_t) = \frac{1}{N^2}\sum_{\boldsymbol{x}_t\in D}\sum_{\boldsymbol{x}_{t'}\in D} k_\sigma(\boldsymbol{x}_t,\boldsymbol{x}_{t'}) = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{I}.$$

式中：$N\times N$ 的矩阵 $\boldsymbol{K}$ 中标号为 (t,t') 的元素等于 $k_\sigma(\boldsymbol{x}_t,\boldsymbol{x}_{t'})$，$\boldsymbol{I}$ 是元素全为 1 的 $N\times 1$ 的列向量.

由此，KECA 数据转换可以表示为：

$$\boldsymbol{\Phi}_e = \boldsymbol{D}_l^{\frac{1}{2}}\boldsymbol{E}_l^{\mathrm{T}}:\min_{\lambda_1,e_1,\cdots,\lambda_N,e_N}\hat{V}(p) - \hat{V}_l(p) = \boldsymbol{D}_l^{\frac{1}{2}}\boldsymbol{E}_l^{\mathrm{T}}:\min_{\lambda_1,e_1,\cdots,\lambda_N,e_N}\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}(\boldsymbol{K}-\boldsymbol{K}_e)\boldsymbol{I}.$$

式中：$\hat{V}_l(p) = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{E}_l\boldsymbol{D}_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{I} = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_e\boldsymbol{I}$，$\boldsymbol{K}_e = \boldsymbol{E}_l\boldsymbol{D}_l\boldsymbol{E}_l^{\mathrm{T}}$.

1.2 KECA 与平均向量的关系

由式(1)，KECA 也可以看作是降维前后核特征空间的数据平均向量的欧几里德长度变化的最小化问题.

$$\hat{V}(p_\sigma) = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{I} = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{I} = \left\langle\frac{1}{N}\sum_{\boldsymbol{x}_i\in D}\boldsymbol{\varphi}(\boldsymbol{x}_i),\frac{1}{N}\sum_{\boldsymbol{x}_{i'}\in D}\boldsymbol{\varphi}(\boldsymbol{x}_{i'})\right\rangle = \|\boldsymbol{m}\|^2. \quad (1)$$

式中：$\boldsymbol{m} = \frac{1}{N}\sum_{\boldsymbol{x}_i\in D}\boldsymbol{\varphi}(\boldsymbol{x}_i)$ 是核特征空间数据集的平均向量，设降维后的数据集为

$$\boldsymbol{\Phi}_e = [\boldsymbol{\Phi}_e(\boldsymbol{x}_1)\quad\boldsymbol{\Phi}_e(\boldsymbol{x}_2)\quad\cdots\quad\boldsymbol{\Phi}_e(\boldsymbol{x}_N)],$$

降维后的熵表示为

$$\hat{V}_k(p_\sigma) = \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_e\boldsymbol{I} = \|\boldsymbol{m}_e\|^2.$$

式中：$\boldsymbol{m}_e = \frac{1}{N}\sum_{\boldsymbol{x}_i\in D}\boldsymbol{\varphi}_e(\boldsymbol{x}_i)$ 是转换后的数据 $\boldsymbol{\Phi}_e$ 的平均向量. 因此 KECA 数据转换还可以用平均向量表示为

$$\boldsymbol{\Phi}_e = \boldsymbol{D}_k^{\frac{1}{2}}\boldsymbol{E}_k^{\mathrm{T}}:\hat{V}(p_\sigma) - \hat{V}_k(p_\sigma) = \boldsymbol{D}_k^{\frac{1}{2}}\boldsymbol{E}_k^{\mathrm{T}}:\min_{\lambda_1,e_1,\cdots,\lambda_N,e_N}\|\boldsymbol{m}\|^2 - \|\boldsymbol{m}_e\|^2.$$

2 核协方差成分分析(KCCA)

最近笔者提出了基于 D-vs-E 的 KCCA 数据转换方法，将其应用于聚类，结果显示 KCCA 方法在对高斯核参数的选择上比 KECA 具有更强的鲁棒性.

2.1 统计量 D-vs-E

通过观察矩阵 $\boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}$，可以建立起 $\hat{V}(p_\sigma)$ 与 $\boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}$ 的等价关系：

$$\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{x}\in D}\hat{p}_\sigma^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \hat{V}(p_\sigma). \quad (2)$$

这引起了笔者对核特征空间中数据集的协方差矩阵 $\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})^{\mathrm{T}}$ 的思考，将式(2)中的 $\boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}$ 替换成协方差矩阵，得到式(3)：

$$\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\left(\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})^{\mathrm{T}}\right)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{x}\in D}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{x}\in D}\frac{1}{N}\sum_{i=1}^{N}k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\mathrm{d}\boldsymbol{x} - \int_{\boldsymbol{x}\in D}\hat{p}_\sigma^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \quad (3)$$

对于高斯核函数，由于 $k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x},\boldsymbol{x}_i) = k_{\sigma/\sqrt{2}}(\boldsymbol{x},\boldsymbol{x}_i)$，所以式(3)可化为

$$\int_{\boldsymbol{x}\in D}\frac{1}{N}\sum_{i=1}^{N}k_{\sigma/\sqrt{2}}(\boldsymbol{x},\boldsymbol{x}_i)\mathrm{d}\boldsymbol{x} - \int_{\boldsymbol{x}\in D}\hat{p}_\sigma^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{x}\in D}\hat{p}_{\sigma/\sqrt{2}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{\boldsymbol{x}\in D}\hat{p}_\sigma^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \quad (4)$$

式中：第 1 项可以近似地表示为

$$\int_{\boldsymbol{x}\in D}\hat{p}_{\sigma/\sqrt{2}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx \sum_{i=1}^{N}\hat{p}_{\sigma/\sqrt{2}}(\boldsymbol{x}_i) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}k_{\sigma/\sqrt{2}}(\boldsymbol{x}_i,\boldsymbol{x}_j),$$

而第 2 项对应于 $\hat{V}(p_\sigma)$，进而对应于 $H(p_\sigma)$，因此式(4)可以近似地表示为

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}k_{\sigma/\sqrt{2}}(\boldsymbol{x}_i,\boldsymbol{x}_j) - \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j). \quad (5)$$

于是式(5)导出了 D-vs-E 的概念，D-vs-E 就是核参数为 $\sigma/\sqrt{2}$ 时的密度总和与熵 $\hat{V}(p_\sigma)$ 的差.

2.2 KCCA 的定义

将式(5)中 2 项的系数提出 $\frac{1}{N}$ 并约去，重写为 $\boldsymbol{I}^{\mathrm{T}}\tilde{\boldsymbol{K}}\boldsymbol{I}$，其中 $\tilde{\boldsymbol{K}}$ 是 $N\times N$ 的矩阵，下标为 (i,j) 的元素

为 $k_{\sigma/\sqrt{2}}(\boldsymbol{x}_i,\boldsymbol{x}_j)-\frac{1}{N}k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)$，设转换后的数据集为

$$\boldsymbol{\Phi}_c=[\boldsymbol{\Phi}_c(\boldsymbol{x}_1)\quad \boldsymbol{\Phi}_c(\boldsymbol{x}_2)\quad \cdots \quad \boldsymbol{\Phi}_c(\boldsymbol{x}_N)],$$

则 KCCA 数据转换定义为

$$\boldsymbol{\Phi}_c=\boldsymbol{D}_k^{\frac{1}{2}}\boldsymbol{E}_k^{\mathrm{T}}:\min_{\lambda_1,e_1,\cdots,\lambda_N,e_N}\boldsymbol{I}^{\mathrm{T}}(\tilde{\boldsymbol{K}}-\tilde{\boldsymbol{K}}_c)\boldsymbol{I}.$$

式中：$\tilde{\boldsymbol{K}}_c=\boldsymbol{\Phi}_c^{\mathrm{T}}\boldsymbol{\Phi}_c=\boldsymbol{E}_k\boldsymbol{D}_k\boldsymbol{E}_k^{\mathrm{T}}$. 需要注意的是 $\tilde{\boldsymbol{K}}$ 可能不是半正定的矩阵，广义 D-vs-E 能够解决这个问题，具体参见文献[16].

2.3　D-vs-E 统计量的优势

可以把式(5)以矩阵形式表示为

$$\frac{1}{N}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma/\sqrt{2}}\boldsymbol{I}-\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}. \tag{6}$$

式中：$\boldsymbol{K}_\sigma$ 表示 $N\times N$ 的矩阵，其下标为(i,j)的元素为 $k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)$. 观察式(6)，D-vs-E 由两部分组成：第 2 部分中$\frac{1}{N}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}$ 是 KECA 中所使用的统计量 $\hat{V}(p)$ 的表达式；第 1 部分是核参数缩小为 $\sigma/\sqrt{2}$时所有样本的密度之和，这是 KECA 方法所不包含的信息. 这说明统计量 D-vs-E 包含了比 $\hat{V}(p)$ 更丰富的信息. 实验也证明了 D-vs-E 不但能像 $\hat{V}(p)$ 一样能很好地提取数据分布的角度结构，而且由于D-vs-E的第 1 项与第 2 项中核参数并不相同，使得在聚类的应用中，能够有效地增强核参数选择的鲁棒性[16]. 其次，观察 KCCA 方法的核矩阵 $\tilde{\boldsymbol{K}}=\boldsymbol{K}_{\sigma/\sqrt{2}}-\boldsymbol{K}_\sigma$，仍是一个 $N\times N$ 矩阵，因此在特征降维中特征分解的过程并没有增加算法的复杂度.

3　高阶统计量

观察式(3)可以看出统计量 D-vs-E 是由数据集的协方差矩阵，即二阶统计量导出的，并且实验证明它比由平均向量导出的 $\hat{V}(p)$ 有更好的性能. 如果使用更高阶的统计量，会不会得到更好的特征降维方法. 将核特征空间中的数据集的协方差矩阵 $\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})(\boldsymbol{\varphi}(\boldsymbol{x}_i)-\boldsymbol{m})^{\mathrm{T}}$用 $\boldsymbol{C}$ 来表示，则有

$$\begin{aligned}\text{D-vs-E}&=\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{C}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&\frac{1}{N}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma/\sqrt{2}}\boldsymbol{I}-\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}.\end{aligned} \tag{7}$$

考虑用 $\boldsymbol{C}^2$ 来代替式(7)中的 $\boldsymbol{C}$，可以导出新的统计量，用 T_2 来表示，于是有

$$\begin{aligned}T_2=&\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{C}^2\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\sum_i^N\sum_j^N\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}-\\&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\sum_i^N\sum_j^N\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}-\\&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\sum_i^N\sum_j^N\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}+\\&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\sum_i^N\sum_j^N\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{8}$$

对于高斯核函数，有 $k_{\sigma_1+\sigma_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\int_{\boldsymbol{x}\in D}k_{\sigma_1}(\boldsymbol{x},\boldsymbol{x}_1)k_{\sigma_2}(\boldsymbol{x},\boldsymbol{x}_2)\mathrm{d}\boldsymbol{x}$，由此可以对式(8)中的每一项分别化简.

第 1 项：

$$\begin{aligned}&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)k_\sigma(\boldsymbol{x}_j,\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&\frac{1}{N^2}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)k_{2\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)=\\&\frac{1}{N^2}\sum_i^N\sum_j^N k_{\frac{2}{\sqrt{5}}\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)=\\&\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I}.\end{aligned} \tag{9}$$

第 2 项：

$$\begin{aligned}&-\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\hat{p}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx\\&-\frac{1}{N^3}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^3}\int_{\boldsymbol{x}\in D}\sum_j^N k_{2\sigma}(\boldsymbol{x},\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^3}\sum_i^N\sum_j^N k_{2\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)=\\&-\frac{1}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.\end{aligned} \tag{10}$$

第 3 项：

$$\begin{aligned}&-\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^2}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\hat{p}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx\\&-\frac{1}{N^3}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_j,\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\\&-\frac{1}{N^3}\sum_i^N\sum_j^N k_{2\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)=\end{aligned}$$

$$-\frac{1}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}. \tag{11}$$

第4项：

$$\frac{1}{N^2}\int_{x\in D}\sum_i^N\sum_j^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^2}\int_{x\in D}\sum_i^N\sum_j^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)p(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$\frac{1}{N^3}\int_{x\in D}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^2}\int_{x\in D}\sum_i^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^2}\sum_i^N\sum_j^N k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m} =$$
$$\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\left(\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\right) =$$
$$\frac{1}{N^4}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2. \tag{12}$$

将式(9)～(12)代入式(8)，得到

$$T_2 \approx \frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I} - \frac{2}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} + \frac{1}{N^4}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2. \tag{13}$$

同理，用 $\boldsymbol{C}^3$ 来代替式(7)中的 $\boldsymbol{C}$，可以导出新的统计量：

$$T_3 \approx \frac{1}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{3}{\sqrt{10}}\sigma}\boldsymbol{I} - \frac{1}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} - \frac{2}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} +$$
$$\frac{2}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} + \frac{1}{N^5}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2 - \frac{1}{N^6}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^3; \tag{14}$$

用 $\boldsymbol{C}^4$ 来代替式(7)中的 $\boldsymbol{C}$，可以导出：

$$T_4 \approx \frac{1}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{4}{\sqrt{17}}\sigma}\boldsymbol{I} - \frac{4}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I} + \frac{3}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} +$$
$$\frac{3}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} - \frac{2}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} -$$
$$\frac{2}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^3 + \frac{1}{N^8}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^4. \tag{15}$$

观察式(13)，它由3个部分组成，第1部分 $\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I}$ 的系数 $\frac{1}{N^2}$ 在3项中最大，因此其在 T_2 中所占比重最大，$\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I}$ 在降维前后的变化则成为了 T_2 降维前后变化的主要影响因素. 既然如此，就只需考虑 T_2 的第1部分 $\frac{1}{N^2}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I}$，后面2个部分都可以略去，这样就达到了简化问题的目的. 再观察式(14)和(15)，同样地，第1部分 $\frac{1}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{3}{\sqrt{10}}\sigma}\boldsymbol{I}$ 和 $\frac{1}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{4}{\sqrt{17}}\sigma}\boldsymbol{I}$ 占统计量 T_3 和 T_4 的比重最大，它们各自的变化对 T_3 和 T_4 的变化影响也最大，与 T_2 同样的道理，在用 T_3 和 T_4 降维的过程中也可以只考虑对统计量影响最大的第1项.

再比较 T_2 中的 $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{2}{\sqrt{5}}\sigma}\boldsymbol{I}$，$T_3$ 中的 $\frac{1}{N^3}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{3}{\sqrt{10}}\sigma}\boldsymbol{I}$ 和 T_4 中的 $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{4}{\sqrt{17}}\sigma}\boldsymbol{I}$，它们的核参数分别为 $\frac{2}{\sqrt{5}}\sigma$、$\frac{3}{\sqrt{10}}\sigma$ 和 $\frac{4}{\sqrt{17}}\sigma$，都小于 σ. 另外可以计算出 T_5、T_6 和 T_7 中比重最大的一项分别为 $\frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{5}{\sqrt{26}}\sigma}\boldsymbol{I}$、$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{6}{\sqrt{37}}\sigma}\boldsymbol{I}$ 和 $\frac{1}{N^7}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{7}{\sqrt{50}}\sigma}\boldsymbol{I}$. 因此可以总结归纳出 T_n 中比重最大的一项为 $\frac{1}{N^n}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{n}{\sqrt{n^2+1}}\sigma}\boldsymbol{I}$，其中 n 是大于0的偶数，从中可以看出两点：

1) $\frac{1}{\sqrt{2}}\sigma < \frac{n}{\sqrt{n^2+1}}\sigma < \sigma$;

2) $\lim\limits_{n\to\infty}\left(\frac{1}{N^n}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{n}{\sqrt{n^2+1}}\sigma}\boldsymbol{I}\right) = 0$.

从第1点可以看出，这些比较重要项的核参数不能随意取，而是在 $\left(\frac{1}{\sqrt{2}}\sigma,\sigma\right)$ 之间；从第2点可以看出，越高阶的统计量提供的信息越少. 因此接近二阶的统计量比较有用，即接近 $\frac{1}{\sqrt{2}}\sigma$ 且大于 $\frac{1}{\sqrt{2}}\sigma$ 的项比较重要. 于是结合统计量 $\hat{V}(p_\sigma)$ 与 D-vs-E，可以构造这样一个统计量：

$$T = \mu(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma/\alpha_1}\boldsymbol{I} + \boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma/\alpha_2}\boldsymbol{I} + \cdots + \boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma/\alpha_n}\boldsymbol{I}) +$$
$$\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I} = \mu\sum_{i=1}^N\sum_{j=1}^N(k_{\sigma/\alpha_1}(\boldsymbol{x}_i,\boldsymbol{x}_j) + k_{\sigma/\alpha_2}(\boldsymbol{x}_i,\boldsymbol{x}_j) + \cdots +$$
$$k_{\sigma/\alpha_n}(\boldsymbol{x}_i,\boldsymbol{x}_j)) + \sum_{i=1}^N\sum_{j=1}^N k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j). \tag{16}$$

式中：$\mu>0$ 是一个调整系数，$\sqrt{2}>\alpha_1,\alpha_2,\cdots,\alpha_n>1$，由第2点可以看出 n 是一个较小的正整数. 式(16)与文献[16]中的广义 D-vs-E 不谋而合，只是在范围的选择($\alpha_1,\alpha_2,\cdots,\alpha_n>1$)上又加以限制. 因此可以通过调节广义 D-vs-E 中的参数 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 来达到整合高阶统计量的目的，同时又避免了大量的计算. 因此，在此类基于 Parzen 窗的特征降维方法中，无需研究更高阶的统计量.

4 结束语

本文对由 R. Jenssen 提出的 KECA 方法和笔者最近提出的 KCCA 方法导出的一系列高阶统计量进行了研究，发现通过调节广义 D-vs-E 中的参数能够

整合高阶统计量，使得广义 D-vs-E 具有了更广泛的意义. 因此在今后的此类基于 Parzen 窗的特征降维方法中无需研究更多计算量的高阶统计量.

附录 A　T_4 推导过程

$$T_4 = \int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{C}^4 \boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} -$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} +$$

$$\frac{1}{N^4}\int_{x \in D} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}) \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

对每一项进行化简：

第 1 项为

$$\frac{1}{N^4}\int_{x \in D} \sum_i^N \sum_j^N \sum_m^N \sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^4}\int_{x \in D} \sum_i^N \sum_j^N \sum_m^N \sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)k_\sigma(\boldsymbol{x}_j,\boldsymbol{x}_m)k_\sigma(\boldsymbol{x}_m,\boldsymbol{x}_n)k_\sigma(\boldsymbol{x}_n,\boldsymbol{x}) \mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^4}\int_{x\in D}\sum_{n}^{N}k_{4\sigma}(\boldsymbol{x},\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^4}\int_{x\in D}\sum_{n}^{N}k_{\frac{4}{\sqrt{17}}\sigma}(\boldsymbol{x},\boldsymbol{x}_n)\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{4}{\sqrt{17}}\sigma}\boldsymbol{I}.$$

第 2 项为

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)k_{\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\int_{x\in D}\sum_{n}^{N}k_{4\sigma}(\boldsymbol{x},\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$

$$-\frac{1}{N^5}\int_{x\in D}\sum_{n}^{N}k_{4\sigma}(\boldsymbol{x},\boldsymbol{x}_n)\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I}.$$

第 3 项为

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\sum_{m}^{N}\sum_{n}^{N}k_{4\sigma}(\boldsymbol{x}_n,\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n) \approx$$

$$-\frac{1}{N^5}\sum_{m}^{N}\sum_{n}^{N}k_{4\sigma}(\boldsymbol{x}_n,\boldsymbol{x}_m) =$$

$$-\frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I}.$$

第 4 项为

$$\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{x\in D}\sum_{m}^{N}k_{3\sigma}(\boldsymbol{x},\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$

$$\frac{1}{N^4}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{x\in D}\sum_{m}^{N}k_{3\sigma}(\boldsymbol{x},\boldsymbol{x}_m)\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I}.$$

第 5 项为

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\int_{x\in D}\sum_{i}^{N}\sum_{j}^{N}\sum_{m}^{N}\sum_{n}^{N}k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)k_{\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$

$$-\frac{1}{N^4}\sum_{m}^{N}\sum_{j}^{N}k_{4\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m) \approx$$

$$-\frac{1}{N^5}\sum_m^N\sum_j^N k_{4\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_j) =$$
$$-\frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I}.$$

第 6 项为

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)k_\sigma(\boldsymbol{x}_m,\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$\frac{1}{N^6}\int_{\boldsymbol{x}\in D}\sum_j^N\sum_m^N\sum_n^N k_{2\sigma}(\boldsymbol{x},\boldsymbol{x}_j)k_\sigma(\boldsymbol{x}_m,\boldsymbol{x}_n)\mathrm{d}(\boldsymbol{x}) =$$
$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.$$

第 7 项为

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)k_\sigma(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$\frac{1}{N^4}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\sum_j^N\sum_n^N k_{3\sigma}(\boldsymbol{x}_n,\boldsymbol{x}_j) =$$
$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I}.$$

第 8 项为

$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_i)\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$-\frac{1}{N^2}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^2\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_\sigma(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$-\frac{1}{N^3}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^2\int_{\boldsymbol{x}\in D}\sum_j^N k_{2\sigma}(\boldsymbol{x},\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x} =$$
$$-\frac{1}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.$$

第 9 项为

$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)k_\sigma(\boldsymbol{x}_j,\boldsymbol{x}_m)k_\sigma(\boldsymbol{x}_m,\boldsymbol{x}_n)k_\sigma(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$-\frac{1}{N^5}\sum_i^N\sum_j^N k_{4\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_i) =$$
$$-\frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I}.$$

第 10 项为

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =$$
$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)k_\sigma(\boldsymbol{x}_j,\boldsymbol{x}_m)k_\sigma(\boldsymbol{x}_m,\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx$$
$$\frac{1}{N^6}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_n^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)k_{2\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_n)\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.$$

第 11 项为

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=$$

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx$$

$$\frac{1}{N^6}\sum_i^N\sum_j^N\sum_m^N\sum_n^N k_{2\sigma}(\boldsymbol{x}_n,\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)=$$

$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.$$

第 12 项为

$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_j)\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=$$

$$-\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx$$

$$-\frac{1}{N^5}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_j,\boldsymbol{x}_m)\mathrm{d}\boldsymbol{x}=$$

$$-\frac{1}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I})^3.$$

第 13 项为

$$\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=$$

$$\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_m^N\sum_n^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)k_{\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx$$

$$\frac{1}{N^4}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\sum_m^N\sum_i^N k_{3\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_i)=$$

$$\frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I}.$$

第 14 项为

$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_m)\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=$$

$$-\frac{1}{N^3}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_m^N\sum_n^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)k_{\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx$$

$$-\frac{1}{N^5}\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_m^N\sum_n^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)k_{\sigma}(\boldsymbol{x}_m,\boldsymbol{x}_n)\mathrm{d}\boldsymbol{x}=$$

$$-\frac{1}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I})^3.$$

第 15 项为

$$-\frac{1}{N^4}\int_{\boldsymbol{x}\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x}_n)\boldsymbol{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=$$

$$-\frac{1}{N^2}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^2\int_{\boldsymbol{x}\in D}\sum_i^N\sum_n^N k_{\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)k_{\sigma}(\boldsymbol{x}_n,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\approx$$

$$-\frac{1}{N^3}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^2\sum_i^N\sum_n^N k_{2\sigma}(\boldsymbol{x}_n,\boldsymbol{x}_i)=$$

$$-\frac{1}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\sigma}\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}.$$

第 16 项为

$$\frac{1}{N^4}\int_{x\in D}\sum_i^N\sum_j^N\sum_m^N\sum_n^N \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_j)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_m)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x}_n)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^3\int_{x\in D}\sum_i^N k_\sigma(\boldsymbol{x},\boldsymbol{x}_i)\boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx$$

$$\frac{1}{N^2}(\boldsymbol{m}^{\mathrm{T}}\boldsymbol{m})^3\int_{x\in D}\sum_i^N k_{2\sigma}(\boldsymbol{x},\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x} =$$

$$\frac{1}{N^8}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^4.$$

将各项带入,化简得

$$T_4 \approx \frac{1}{N^4}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{4}{\sqrt{17}}\sigma}\boldsymbol{I} - \frac{1}{N^5}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I} + \frac{3}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} + \frac{3}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} - \frac{2}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} - \frac{2}{N^7}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^3 + \frac{1}{N^8}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^4.$$

附录 *B*　T$_6$ 推导结果

$$T_6 \approx \frac{1}{N^6}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{\frac{6}{\sqrt{37}}\sigma}\boldsymbol{I} - \frac{6}{N^7}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{6\sigma}\boldsymbol{I} + \frac{5}{N^8}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{5\sigma}\boldsymbol{I} + \left(\frac{5}{N^8} - \frac{1}{N^9}\right)\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I} + \frac{5}{N^8}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} + \left(\frac{1}{N^{10}} - \frac{8}{N^9}\right)(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} - \frac{3}{N^9}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{4\sigma}\boldsymbol{I} - \frac{4}{N^9}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^2\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} - \frac{4}{N^9}\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I})^2 + \frac{9}{N^{10}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^3\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} + \frac{3}{N^{10}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^4 - \frac{4}{N^{11}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^5 + \frac{2}{N^{10}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^3\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{3\sigma}\boldsymbol{I} - \frac{2}{N^{11}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^4\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_{2\sigma}\boldsymbol{I} + \frac{1}{N^{12}}(\boldsymbol{I}^{\mathrm{T}}\boldsymbol{K}_\sigma\boldsymbol{I})^6.$$

参考文献:

[1]LU Wei, SUN Wei, CHUNG Fulai, et al. Revealing digital fakery using multiresolution decomposition and higher order statistics[J]. Engineering Applications of Artificial Intelligence, 2011, 24(4): 666-672.

[2]MUNSHI D, KITCHING T, HEAVENS A, et al. Higher order statistics for three-dimensional shear and flexion[J]. Monthly Notices of the Royal Astronomical Society, 2011, 416(3): 1629-1653.

[3]NAPOLITANO A, TESAURO M. Almost-periodic higher order statistic estimation[J]. IEEE Transactions on Information Theory, 2011, 57(1): 514-533.

[4]KALIDINDI S, NIEZGODA S, SALEM A. Microstructure informatics using higher-order statistics and efficient data-mining protocols[J]. JOM, 2011, 63(4): 34-41.

[5]AGUERA-PEREZ A, PALOMARES-SALAS J, De LA ROSA J, et al. Characterization of electrical sags and swells using higher-order statistical estimators[J]. Measurement, 2011, 44(8): 1453-1460.

[6]LABBI A, BOSCH H, PELLEGRINI C. High order statistics for image classification[J]. International Journal of Neural Systems, 2001, 11(4): 371-378.

[7]张丽琴,詹麒,朱培民,等.地震散射波场的高阶统计分析[J].石油地球物理勘探, 2004, 39(1): 45-49.
ZHANG Liqin, ZHAN Qi, ZHU Peimin, et al. High-order statistic analysis of seismic dispersive wavefield[J]. Oil Geophysical Prospecting, 2004, 39(1): 45-49.

[8]ANASTASSIOU G, DUMAN O. High order statistical fuzzy Korovkin theory[J]. Stochastic Analysis and Applications, 2009, 27(3): 543-554.

[9]NIKORA V, GORING D. Martian topography: scaling, craters, and high-order statistics[J]. Mathematical Geology, 2005, 37(4): 337-355.

[10]PORAT B, FRIEDLANDER B. Direction finding algorithms based on high-order statistics[J]. IEEE Transactions on Signal Processing, 1991, 39(9): 2016-2024.

[11]COURNAPEAU D, KAWAHARA T. Voice activity detection based on high order statistics and online EM algorithm[J]. IEICE Transactions on Information and Systems, 2008, 91(12): 2854-2861.

[12]REN Hsuan, DU Qian, WANG Jing, et al. Automatic target recognition for hyperspectral imagery using high-order statistics[J]. IEEE Transactions on Aerospace and Electronic Systems, 2006, 42(4): 1372-1385.

[13]TAOUFIKI M, ADIB A, ABOUTAJDINE D. Blind separation of any source distributions via high-order statistics[J]. Signal Processing, 2007, 87(8): 1882-1889.

[14]YUAN Jinghe, HU Ziqiang. High-order statistical blind deconvolution of spectroscopic data with a Gauss-Newton algorithm[J]. Applied Spectroscopy, 2006, 60(6): 692-697.

[15] JENSSEN R. Kernel entropy component analysis [J]. IEEE Transactions on Pattern Analysis and Machine Intelligence, 2010, 32(5): 847-860.

[16] 闫晓波,王士同,郭慧玲. 核协方差成分分析方法及其在聚类中的应用[J]. 计算机科学, 2012, 39(9): 229-234.
YAN Xiaobo, WANG Shitong, GUO Huiling. Kernel covariance component analysis and its application in clustering[J]. Computer Science, 2012, 39(9): 229-234.

[17] RENYI A. On measures of entropy and information[C]// Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability. Berkeley, USA, 1961: 547-561.

[18] PARZEN E. On estimation of a probability density function and mode [J]. The Annals of Mathematical Statistics, 1962, 33(3): 1065-1076.

[19] DENG Zhaohong, CHUNG Fulai, WANG Shitong. FRSDE: fast reduced set density estimator using minimal enclosing ball approximation [J]. Pattern Recognition, 2008, 41(4): 1363-1372.

[20] KOLLIOS G, GUNOPULOS D, KOUDAS N, et al. Efficient biased sampling for approximate clustering and outlier detection in large data sets [J]. IEEE Transactions on Knowledge and Date Engineering, 2003, 15(5): 1170-1187.

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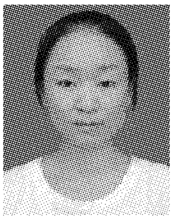

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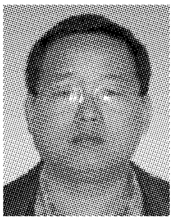

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